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Bending of large curvature beams. I. Stress method approach

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Abstract

The paper presents a coherent theory of the *uniform bending* problem in a circular curved beam, with multi-connected cross-section, having a large radius of curvature with respect to its width. The three-dimensional elastic problem is solved, in the case of linear homogeneous isotropic body, assuming the stress tensor as the unknown and by exactly satisfying the field compatibility equations.

The mathematical structure of the governing boundary value problem (BVP), enlightened here for the first time, is unexpectedly complicated: a *fourth-order* elliptic (variable coefficients) partial differential equation with two *degenerate unstable* boundary conditions (i.e. involving second and third order partial derivatives in a direction that becomes tangent at several points of the boundary).

Such a kind of BVP seems to be typical of the curved beam bending problem since it also appears in the displacement approach (Mentrasti, 2001. part II, Int. J. Solids Struct. 38, 5727–5745).

As a final point, it is reduced to a simpler problem by an ad hoc integral representation, assuming the ρ-convexity of the cross-section domain. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Curved beams, with large radius of curvature with respect to the width, subjected to bending moments, have been studied widely in the past under very restrictive hypotheses regarding the geometry of the cross-section, the displacement field or the state of stress: among the others, the formulations by Golovin (1880–1881) quoted by Timoshenko (1953), Fubini (1937), Southwell (1942), and Freiberger and Smith (1949) can be cited. Particularly interesting is the paper by Mitchell (1899) in which, following a displacement approach, a fourth-order field equation appears for the first time with two boundary conditions involving second and third derivatives.

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Nomenclature

$\mathscr{A}$	cross-section domain, N-connected, with boundary $\partial\mathscr{A}$
$\mathscr{A}_z$	projection of $\mathscr{A}$ on the z-axis
$\mathscr{E}^2$, $\mathscr{E}^3$	Euclidean plane, space
$\mathrm{Int}(\mathscr{L})$	simply connected domain enclosed within the curve $\mathscr{L}$
$c_i \subset \partial\mathscr{A}$	ith regular curves of the boundary, $i = 1 \ldots N$
c_0, c_0^*	bending constant, $c_0^* := \nu/(1-\nu^2)c_0$
H	generalized potential stress function
E, G, ν	elastic constants
$n := [n_\rho \quad n_Z]^{\mathrm{T}}$	components of the outward normal, $\boldsymbol{n}$, at $\partial\mathscr{A}$
$t := [-n_Z \quad n_\rho]^{\mathrm{T}}$	components of the tangent unit vector, $\boldsymbol{t}$, at $\partial\mathscr{A}$
$q := 1/q\,[n_Z \quad \nu n_\rho]^{\mathrm{T}}$	components of the oblique unit vector at $\partial\mathscr{A}$, $q := (\nu^2 n_\rho^2 + n_z^2)^{1/2}$
φ	longitude of the current cross-section
σ_{ij}, ϵ_{ij}	stress and strain components in a cylindrical system of co-ordinate
ψ	potential stress function
$\Lambda := \rho\partial_\rho(1/\rho)\partial_\rho + \partial_{zz}$	field operator, Eq. (5.6)
$\{O, \rho, z\}$	system of co-ordinates in the plane of the cross-section
$\{O, \rho, \varphi, z\}$	cylindrical system of co-ordinates
$\{O, X, Y, Z\}$	global system of co-ordinates (Z is the axis of revolution)
$(\ )_{,\xi}$	partial derivative with respect to the variable ξ
$\square$	end of a proof

The beam with rectangular cross-section under plane state of stress has been studied recently by Chianese and Erdlach (1988), and Kardomateas (1991). Furthermore, the relationships between the Timoshenko and Euler–Bernoulli formulations is recently presented by Lim et al. (1997) for the same cross-section.

Three-dimensional (3D) elastic problems were solved numerically for a trapezoid cross-section (Cook, 1989) and by means of ad hoc formulations for rectangular box beams (Cook, 1991; De Melo and Vaz, 1992). For a circular torus, an analytical solution was given in the fundamental paper by Sadowsky and Sternberg (1953) in which a conjecture about the functional form of the displacement field was assumed, following Mitchell: the governing boundary value problem (BVP) is a differential system of two second order homogeneous partial differential equations (PDEs) in two unknown functions, coupled by two boundary conditions involving second order partial derivatives.

Despite the importance of the problem, both for its *intrinsic interest* and the possibility of implementing *validation procedures* in computational mechanics (general 3D beam theories, small deformation superimposed to large bending states, bending delamination, etc.), a complete and coherent formulation seems to be lacking.

The aim of this paper is to solve *exactly* the 3D elastic problem assuming the stress tensor as an unknown (function of only *two* variables in the plane of the cross-section): in a first step a potential function is used to satisfy the equilibrium equations; then, some of the remaining significant compatibility equations are reduced to a *fourth-order elliptic* PDE with variable coefficients. The equilibrium boundary conditions (BCs) lead to a pair of equations in the second and third partial derivatives of the unknown function: *degenerate unstable* BCs.

This kind of BVP, presented here for the *first time*, does not seems to be connected to the formulation adopted, but is peculiar to the bending problem. The companion paper (Part II: Mentrasti, 2000), attacking the problem with a displacement approach, leads to analogous conclusions.

The degenerate unstable BVP is finally reduced to a simpler problem by an ad hoc integral representation of the unknown function, assuming the ρ-convexity of the cross-section domain. In the case of rectangular cross-section the solvability of the problem is shown explicitly.

2. Equilibrium equation

2.1. The body

Let $\mathscr{A}$ be the closure of a N-connected open set of the Euclidean plane $\mathscr{E}^2$, bounded by regular curves $c_i \subset \partial\mathscr{A}$, $i = 1 \dots N$, where c_1 is the outermost boundary (Fig. 2); this domain is described in a local system of co-ordinates $\{O, \rho, z\}$. The distance of $\mathscr{A}$ from the z-axis is strictly positive and it is of the same order of magnitude of its diameter (*large curvature beam*).

The beam $\mathscr{B} \subset \mathscr{E}^3$ is a toroid, i.e. a body generated by the revolution of its plane *cross-section* $\mathscr{A}$ about the Z-axis of a global system of orthonormal co-ordinates $\{O, X, Y, Z\}$, with $z \equiv Z$. $\mathscr{B}$ is devised as $\mathscr{A} \times \Upsilon$, where $\Upsilon := \{\varphi \in \mathscr{R} \,|\, 0 \leqslant \varphi \leqslant \varphi_0\}$ is the solid angle enclosing the beam, φ being the longitude of the current cross-section measured from the X-axis as shown in Fig. 1.

This *body* is linearly elastic, homogeneous and isotropic; isotherm condition is assumed.

The mantle $\partial\mathscr{A} \times \Upsilon$ of the beam is not loaded, while on the base $\mathscr{A}_{\varphi_0}(\mathscr{A}_0)$ a bending moment $m_Z(-m_Z)$ is applied. Body forces are nil.

2.2. Uniform bending

The beam is assumed to be subjected to a uniform bending, in the sense of the definition below.

The physical motivation of this hypothesis is to disregard the pointwise stress distribution on the bases and to deal only with the resultant force of the applied stress (*relaxed traction boundary condition*).

The *non-uniform* case is also possible, but it is not considered in this paper.

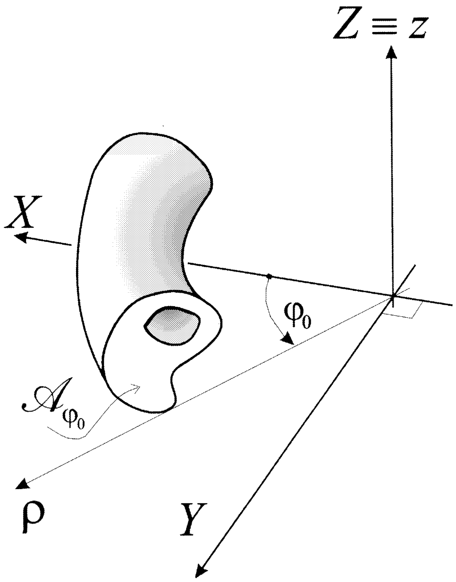

Fig. 1. Curved beam: global and local system of reference.

Definition. A curved beam is in a *uniform* thermo-mechanical *state* if every *local definable* quantity (e.g. stress, strain, constitutive law, entropy, temperature, etc.) is *invariant* under the *group* $\mathscr{G}_Z$ of rigid rotations about Z.

From a mechanical point of view this functional conjecture means that the behaviour of the beam, at each cross-section, is *locally* indiscernible from any other cross-section.

The *analytical consequence* of uniform bending is that the stress and strain tensors are independent of the φ co-ordinate (not so, of course, is the displacement field).

2.3. Bending/shear–torsion uncoupling

The field equilibrium equations, in cylindrical co-ordinates, reduce therefore to the following system of PDEs in the ρ and z variables only:

$$\frac{1}{\rho}(\rho\sigma_\rho)_{,\rho} + \tau_{\rho z,z} - \frac{1}{\rho}\sigma_\varphi = 0, \tag{2.1}$$

$$\frac{1}{\rho^2}(\rho^2\tau_{\rho\varphi})_{,\rho} + \tau_{\varphi z,z} = 0, \tag{2.2}$$

$$\frac{1}{\rho}(\rho\tau_{\rho z})_{,\rho} + \sigma_{z,z} = 0. \tag{2.3}$$

Stress BCs are

$$\sigma_\rho n_\rho + \tau_{\rho z} n_z = 0, \tag{2.4}$$

$$\tau_{\rho\varphi} n_\rho + \tau_{\varphi z} n_z = 0, \tag{2.5}$$

$$\tau_{\rho z} n_\rho + \sigma_z n_z = 0, \tag{2.6}$$

where $\boldsymbol{n}$ is the outward normal at $\partial\mathscr{A}$.

This differential system can be split into two formally uncoupled subsystems because the $\tau_{\rho\varphi}$ and $\tau_{\varphi z}$ components appear in Eqs. (2.2) and (2.5) only.

When the constitutive equation is such that $\tau_{\rho\varphi}$ and $\tau_{\varphi z}$ depend on $\gamma_{\rho\varphi}$ and $\gamma_{\varphi z}$ only the problem is simplified (as it occurs for the isotropic body, explicitly discussed in the next sections).

In fact, in this case Eqs. (2.2) and (2.5) govern the *shear–torsion* problem, as shown by Mentrasti (1996), and $\tau_{\rho\varphi}$ and $\tau_{\varphi z}$ will be therefore ignored in the sequel (*bending/shear–torsion uncoupling hypothesis*).

2.4. General solution of the equilibrium equations

The significant equations are satisfied in two steps: first, Eq. (2.3) is read as the (necessary) condition of existence for a potential function $\psi(\rho, z)$, such that

$$\tau_{\rho z} = \frac{1}{\rho}\psi_{,z}, \tag{2.7}$$

$$\sigma_z = -\frac{1}{\rho}\psi_{,\rho} \tag{2.8}$$

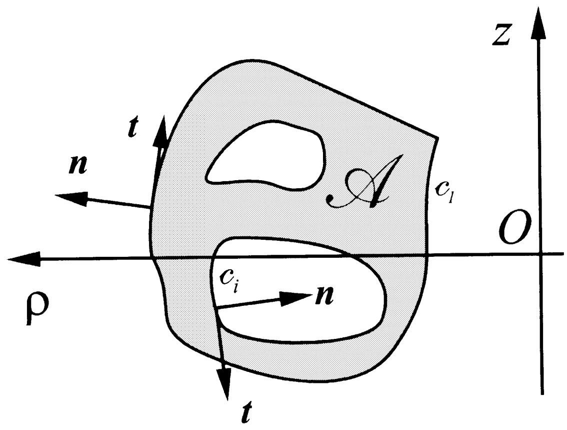

Fig. 2. Cross-section in the ρ–z plane.

subject to the (sufficient) monodromy conditions

$$\oint_{c_i} \mathrm{d}_t\psi = 0, \tag{2.9}$$

along each curve c_i constituting the boundary $\partial\mathscr{A}$ (Fig. 2).

Then, Eq. (2.1) is solved for $\sigma_\varphi(\rho, z)$ giving

$$\sigma_\varphi = (\rho\,\sigma_\rho)_{,\rho} + \psi_{,zz}. \tag{2.10}$$

By using the tangent unit vector $\boldsymbol{t}$, $t := [-n_z \quad n_\rho]^{\mathrm{T}}$, the BC (2.6) can be written

$$0 = \frac{1}{\rho}(\psi_{,z}n_\rho - \psi_{,\rho}n_z) = \frac{1}{\rho}(\psi_{,z}t_z + \psi_{,\rho}t_\rho) = \frac{1}{\rho}\psi_{,s} \tag{2.11}$$

that, after integration along $\partial\mathscr{A}$, gives

$$\psi(\mathbf{x}) = \psi_i^0, \quad \forall\mathbf{x} \in c_i. \tag{2.12}$$

Observe that when these conditions are satisfied, the monodromy conditions (2.9) are automatically fulfilled (if $\mathscr{A}$ is simply connected, the unique constant ψ_1^0 can be set to zero, as will be widely discussed in the sequel).

The remaining BC, Eq. (2.4), becomes

$$\sigma_\rho n_\rho + \frac{1}{\rho}\psi_{,z}n_z = 0. \tag{2.13}$$

In this way, ψ and σ_ρ are the *natural unknowns* of the problem.

3. Field compatibility equations

The above components of the stress tensor fulfill the field equilibrium equations identically. In addition, they must verify the six compatibility conditions (A.1)–(A.6), shown in Appendix A.

In the present case Eqs. (A.4) and (A.5) are identically satisfied and the others become

$$I := -\varepsilon_{\varphi,zz} + \frac{2}{\rho}\varepsilon_{\rho z,z} - \frac{1}{\rho}\varepsilon_{z,\rho} = 0, \tag{3.1}$$

$$II := 2\varepsilon_{\rho z,\rho z} - \varepsilon_{\rho,zz} - \varepsilon_{z,\rho\rho} = 0, \tag{3.2}$$

$$III := -\varepsilon_{\varphi,\rho\rho} + \frac{1}{\rho}\varepsilon_{\rho,\rho} - \frac{2}{\rho}\varepsilon_{\varphi,\rho} = 0, \tag{3.3}$$

$$VI := \varepsilon_{\varphi,\rho z} - \frac{1}{\rho}(\varepsilon_{\rho} - \varepsilon_{\varphi})_{,z} = 0. \tag{3.4}$$

in which I (II, etc.) denotes the left-hand side of Eq. (3.1) (Eq. (3.2), etc.), regarded as a function of $\varepsilon_{ij}(\rho, z)$.

3.1. Reduction of the significant equations

Eq. (3.1) can be arranged as

$$2\varepsilon_{\rho z,z} - \varepsilon_{z,\rho} = \rho\varepsilon_{\varphi,zz} \tag{3.5}$$

and Eq. (3.2) as

$$(2\varepsilon_{\rho z,z} - \varepsilon_{z,\rho})_{,\rho} = \varepsilon_{\rho,zz}; \tag{3.6}$$

then by substituting the former relationship in the latter, results in

$$[(\rho\varepsilon_{\varphi})_{,\rho} - \varepsilon_{\rho})]_{,zz} = 0. \tag{3.7}$$

On the other hand Eq. (3.4), $VI = 0$, can be written as

$$[(\rho\varepsilon_{\varphi})_{,\rho} - \varepsilon_{\rho}]_{,z} = 0. \tag{3.8}$$

Therefore, the following conclusion holds true:

Lemma 1. *If* $\mathrm{I} = 0$ *and* $VI = 0$ *then* $\mathrm{II} = (\rho I)_{,\rho} + VI_{,z} = 0$.

By collecting the partial derivatives with respect to ρ, *III* becomes

$$III = -\frac{1}{\rho}[(\rho\varepsilon_{\varphi})_{,\rho\rho} - \varepsilon_{\rho,\rho}] = -\frac{1}{\rho}[(\rho\varepsilon_{\varphi})_{,\rho} - \varepsilon_{\rho}]_{,\rho}. \tag{3.9}$$

Eqs. (3.8) and (3.9) finally lead to the required relationship

$$(\rho\varepsilon_{\varphi})_{,\rho} - \varepsilon_{\rho} = c_0. \tag{3.10}$$

Lemma 2. *Eq.* (3.10) *is equivalent to the system* $\{III = 0, VI = 0\}$.

The final result can be therefore stated:

Theorem 1. *The system of Eqs.* (3.5) and (3.10) *is equivalent to the four significant compatibility equations* $I = 0,\ II = 0,\ III = 0,\ VI = 0$.

4. Governing equation

In order to obtain intelligible relationships in the unknown functions ψ and σ_ρ, the cumbersome algebraic manipulations necessary to substitute the strain expressions in system (3.5)–(3.10), must be carried out with some order.

4.1. Integration of Eq. (3.10)

Let L denote the expression

$$L := \varepsilon_{\varphi,\rho} - \frac{1}{\rho}(\varepsilon_\rho - \varepsilon_\varphi), \tag{4.1}$$

such that Eq. (3.10) can be rewritten, for the sake of brevity, as $\rho L = c_0$.

Using the *linearly elastic*, *homogeneous* and *isotropic* constitutive equations and the results (2.7), (2.8) and (2.10), L is written as

$$L = 3\sigma_{\rho,\rho} + \rho\sigma_{\rho,\rho\rho} + \psi_{,\rho zz} + v\left(\frac{1}{\rho}\psi_{,\rho}\right)_{,\rho} + (1+v)\frac{1}{\rho}\psi_{,zz} \tag{4.2}$$

the Young modulus being included in c_0. By isolating the terms derived with respect to ρ from those derived with respect to z and rearranging the result, Eq. (3.10) finally becomes

$$\left[\frac{1}{\rho}(\rho^2\sigma_\rho + v\psi)_{,\rho}\right]_{,\rho} + \frac{1}{\rho}[(\rho(\psi)_{,\rho} + v\psi]_{,zz} = c_0\frac{1}{\rho}. \tag{4.3}$$

The solution of its *associate homogeneous* PDE is derived in Appendix B, together with a *particular solution* $\sigma_\rho^{\mathrm{P}}(\rho)$, in the following form:

$$\rho^2\sigma_\rho + v\psi = F_{,zz}, \tag{4.4}$$

$$(\rho\psi)_{,\rho} + v\psi = -\rho\left(\frac{1}{\rho}F_{,\rho}\right)_{,\rho}, \tag{4.5}$$

$$\sigma_\rho^{\mathrm{P}}(\rho) := \frac{1}{4}c_0(2\ln\rho - 1) + c_1 + c_2\frac{1}{\rho^2}, \tag{4.6}$$

where $F(\rho, z)$ is a new unknown function, and c_1, c_2 are (unessential) constants whose meaning is explained in Appendix B.

Furthermore, Eq. (4.5) is reckoned as an ordinary differential equation (ODE) in which z appears as a parameter; its solution is derived in Appendix C as:

$$\psi = [(1+v)\phi - \rho\phi_{,\rho}]\rho^{(1-v)}, \tag{4.7}$$

$$\rho^{(1-v)}\phi_{,\rho} = F_{,\rho}. \tag{4.8}$$

4.2. Reduction of the differential system (4.7) and (4.8)

The aim of the last step is (i) to restate the solution of the previous system of PDE in terms of a single unknown function that (ii) satisfies identically the compatibility equation (3.10).

In this scope, ϕ is tentatively devised as a linear combination of a new unknown function H and its partial derivative $H,_{\rho}$ whose coefficients are polynomial in ρ. The attempt succeeded in a surprisingly efficient manner; in fact, posing

$$\phi := \rho^{\nu} H_{,\rho} \tag{4.9}$$

the left hand side of Eq. (4.8) becomes the derivative of an algebraic expression $\rho^{(1-\nu)}\phi_{,\rho} \equiv [(1-\nu)H + \rho H_{,\rho}]_{,\rho}$; consequently the solution of Eq. (4.8) is quickly obtained as

$$F = (1-\nu)H + \rho H_{,\rho} + b(z), \tag{4.10}$$

in which the undetermined function $b(z)$ can be included in H.

5. Governing boundary value problem with degenerate unstable boundary conditions

The main result obtained in the previous paragraphs is the general solution of the compatibility equation (3.10); after appropriate reductions, it can be summarized as follows:

$$\psi := -\rho\left(\frac{1}{\rho}H_{,\rho}\right)_{,\rho}, \tag{5.1}$$

$$\sigma_{\rho} = \nu\frac{1}{\rho}\left(\frac{1}{\rho}H_{,\rho}\right)_{,\rho} + \frac{1}{\rho^2}[(\nu-1)H + \rho H_{,\rho}]_{,zz} + \sigma_{\rho}^{\mathrm{P}}, \tag{5.2}$$

$$\sigma_{\rho}^{\mathrm{P}}(\rho) := \tfrac{1}{4}c_0(2\ln\rho - 1), \tag{5.3}$$

where $H(\rho, z)$ is the ultimate unknown generalized *potential* function.

The remaining condition to fulfill is the compatibility equation $I = 0$.

By expanding the strains as functions of H, through the above expression of ψ and σ_{ρ}, Eq. (3.1) writes

$$\rho(H_{,\rho\rho\rho\rho} + 2H_{,\rho\rho zz} + H_{,zzzz}) - \left[2(H_{,\rho\rho} + H_{,zz}) - \frac{3}{\rho}H_{,\rho}\right]_{,\rho} - \frac{\nu}{1-\nu^2}c_0\rho = 0. \tag{5.4}$$

5.1. Field equation

After an amount of algebraic handling, Eq. (5.4) can be rewritten as

$$\left\{\frac{1}{\rho}\left[\rho\left(\frac{1}{\rho}H_{,\rho}\right)_{,\rho} + H_{,zz}\right]_{,\rho}\right\}_{,\rho} + \left\{\frac{1}{\rho}\left[\rho\left(\frac{1}{\rho}H_{,\rho}\right)_{,\rho} + H_{,zz}\right]_{,z}\right\}_{,z} = \frac{\nu}{1-\nu^2}c_0\frac{1}{\rho}; \tag{5.5}$$

this *governing equation* can be easily read in the following more expressive form

$$\Lambda\Lambda H = c_0^{*}, \quad \forall \mathbf{x} \in \mathscr{A} \tag{5.6}$$

in which the operator Λ is defined as

$$\Lambda \equiv \Lambda_\rho + \Lambda_z := \rho \partial_\rho \frac{1}{\rho} \partial_\rho + \partial_{zz} \tag{5.7}$$

and

$$c_0^* := \frac{v}{1 - v^2} c_0. \tag{5.8}$$

5.2. Boundary conditions

The BCs (2.12), $\psi = \psi_i^0$, using Eq. (5.1), leads to

$$\rho \left(\frac{1}{\rho} H_{,\rho} \right)_{,\rho} = -\psi_i^0, \quad \forall \mathbf{x} \in c_i \subseteq \partial \mathscr{A}. \tag{5.9}$$

The first BC is reduced, by substituting the results (5.1)–(5.3) in Eq. (2.13), to the following form

$$\left[\frac{1}{\rho} H_{,\rho zz} - (1 - v) \frac{1}{\rho^2} H_{,zz} + v \frac{1}{\rho} \left(\frac{1}{\rho} H_{,\rho} \right)_{,\rho} + \sigma_\rho^{\mathrm{P}} \right] n_\rho - \left(\frac{1}{\rho} H_{,\rho} \right)_{,\rho z} n_z = 0. \tag{5.10}$$

Using the previous BC in the third addend, after multiplying by ρ^2, this equation becomes

$$[\rho H_{,\rho zz} - (1 - v) H_{,zz}] n_\rho - \rho^2 \left(\frac{1}{\rho} H_{,\rho} \right)_{,\rho z} n_z + \left(\rho^2 \sigma_\rho^{\mathrm{P}} - \psi_i^0 \right) n_\rho = 0. \tag{5.11}$$

To enlighten the structure of the BVP, this *unstable* condition can be modified in such a way that an *oblique* derivative could appear.

By expanding the last derivative with respect to ρ and recognizing the terms $(-H_{,zz} n_\rho + H_{,\rho z} n_z)$ as $(-H_{,z})_{,t}$ and $(H_{,\rho zz} n_\rho - H_{,\rho\rho z} n_z)$ as $(H_{,\rho z})_{,t}$, respectively, Eq. (5.11) becomes

$$v H_{,zz} n_\rho - (H_{,z})_{,t} + \rho (H_{,\rho z})_{,t} + \left(\rho^2 \sigma_\rho^{\mathrm{P}} - \psi_i^0 \right) n_\rho = 0. \tag{5.12}$$

Consider now the identity $\rho (H_{,\rho z})_{,t} \equiv (\rho H_{,\rho z})_{,t} - H_{,\rho z} t_\rho$; then the previous equation can be rewritten as

$$(H_{,z})_{,\rho} n_z + (H_{,z})_{,z} v n_\rho + (\rho H_{,\rho z} - H_{,z})_{,t} + \left(\rho^2 \sigma_\rho^{\mathrm{P}} - v \psi_i^0 \right) n_\rho = 0. \tag{5.13}$$

When a unit vector $\boldsymbol{q}$, oblique with respect to the normal $\boldsymbol{n}$ at the boundary, is defined with components

$$q := \frac{1}{q} [n_z \quad v n_\rho]^{\mathrm{T}}, \qquad q := (v^2 n_\rho^2 + n_z^2)^{1/2}, \tag{5.14}$$

then the alternative form of the second BC is finally obtained as

$$q (H_{,z})_{,q} + \left[\rho^2 \left(\frac{1}{\rho} H \right)_{,\rho z} \right]_{,t} + \left(\rho^2 \sigma_\rho^{\mathrm{P}} - \psi_i^0 \right) n_\rho = 0, \quad \forall \mathbf{x} \in c_i \subseteq \partial \mathscr{A}. \tag{5.15}$$

Since $\boldsymbol{q} \cdot \boldsymbol{n} = (1 + v) n_\rho n_z / q$, the unit vector $\boldsymbol{q}$ varies from $\boldsymbol{n}$ to $-\boldsymbol{n}$ (it turns counter wards with respect the rotation of $\boldsymbol{n}$ along the boundary of the cross-section as shown in Fig. 3); thus Eq. (5.15) is an *oblique degenerate* (unstable) BC.

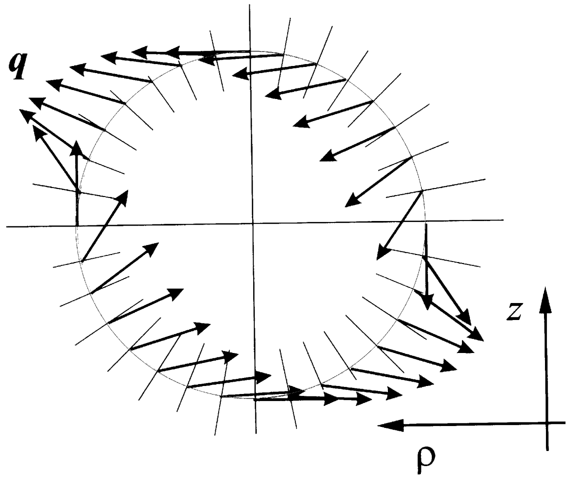

Fig. 3. Degenerate direction $\boldsymbol{q}$, with respect to the normal $\boldsymbol{n}$.

5.3. Discrete compatibility equations

When the cross-section $\mathscr{A}$ is a multi-connected domain, the field compatibility equations discussed in Section 3 are no longer sufficient to determine the displacement field $\boldsymbol{u}$ univocally: the $\boldsymbol{u}$ jump along every closed path $\mathscr{L} \subset \mathscr{A}$ contouring the boundary curve c_i must be zero, that is $\oint_{c_i} \mathrm{d}_t \boldsymbol{u} = 0$.

A complete analysis of the displacement field is carried out in Part II: Section 2 so that only the results strictly necessary to evaluate $\mathrm{d}_t \boldsymbol{u}$ are summarized here:

1. The out of plane displacement v is completely determined by Eq. (2.41) Part II as $v = b_{00}\rho\varphi$ and therefore it does not induce any compatibility condition.
2. From the definition of $\varepsilon_\varphi := (v_{,\varphi} + u)/\rho$, the *continuous* component u can be derived as

$$u = \rho(\varepsilon_\varphi - b_{00}), \tag{5.16}$$

   from the trivial continuity of $\varepsilon_\varphi[\sigma_{ij}(H)]$ and v.
3. Since $\gamma_{\rho z} := w_{,\rho} + u_{,z}$, using the previous relationship and considering that $v_{,z} = 0, w_{,\rho}$ can be obtained as

$$w_{,\rho} = \gamma_{\rho z} - \rho\varepsilon_{\varphi,z}. \tag{5.17}$$

4. Finally $\varepsilon_z := w_{,z}$, or

$$w_{,z} = \varepsilon_z. \tag{5.18}$$

Thus the required monodromy conditions involve no more than the w component; furthermore, by employing the last results $\mathrm{d}_t w$ can be written as a function of the strain components

$$\mathrm{d}_t w = [(\gamma_{\rho z} - \rho\varepsilon_{\varphi,z})t_\rho + \varepsilon_z t_z]\,\mathrm{d}l; \tag{5.19}$$

consequently, it is easy to write $\mathrm{d}_t w$ as an explicit function of $H(\rho, z)$ using relationships (2.7), (2.8) and (2.10), the isotropic constitutive equation and Eqs. (5.1)–(5.3):

$$\frac{E}{(1+v)}\mathrm{d}_t w = \left\{ \left[(v-2)\left(\frac{1}{\rho}H_{,\rho}\right)_{,\rho} - (1-v)\frac{1}{\rho}H_{,zz} \right]_{,z} t_p + \frac{1}{\rho}\left[-vH_{,\rho zz} + (1-v)\left(\rho\left(\frac{1}{\rho}H_{,\rho}\right)_{,\rho}\right)_{,\rho} - \frac{v}{1+v}c_0 \ln\rho \right] t_z \right\} \mathrm{d}l. \tag{5.20}$$

It is worth mentioning briefly that the *necessary condition* for this differential form to be independent of the integration path is precisely the field equation $\Lambda\Lambda H = c_0^*$, previously derived by fulfilling the general 3D compatibility condition!

To render the expression enclosed in curly brackets decipherable, several algebraic manipulations are required:

1. Extract from the first addend the term $(v-1)(1/\rho)[\rho((1/\rho)H_{,\rho})_{,\rho}]_{,z}t_\rho$ and associate it with $(1-v)(1/\rho)(\rho((1/\rho)H_{,\rho})_{,\rho})_{,\rho}t_z$ to obtain $(1-v)(1/\rho)[\rho((1/\rho)H_{,\rho})_{,\rho}]_{,n}$.
2. Proceed analogously with terms $-(1-v)(1/\rho)H_{,zzz}t_\rho$ and $(1-v)(1/\rho)H_{,\rho zz}t_z$ to give $(1-v)(1/\rho)[H_{,zz}]_{,n}$.
3. Finally, consider the remaining first addend $-((1/\rho)H_{,\rho})_{,\rho z}t_\rho$ and the term $-(1/\rho)H_{,\rho zz}$ rising from step 2; they can be collected in $-((1/\rho)H_{,\rho z})_{,t}$.

Accordingly, Eq. (5.20) can be rewritten as

$$\begin{aligned}\frac{E}{(1+v)}\mathrm{d}_t w &= \left\{ (1-v)\frac{1}{\rho}\left[\rho\left(\frac{1}{\rho}H_{,\rho}\right)_{,\rho}\right]_{,n} + (1-v)\frac{1}{\rho}[H_{,zz}]_{,n} - \left(\frac{1}{\rho}H_{,\rho z}\right)_{,t} - \frac{v}{1+v}c_0 \ln\rho n_\rho \right\} \mathrm{d}l \\ &= (1-v)\left\{ \frac{1}{\rho}(\Lambda H)_{,n} - \frac{v}{1-v^2}c_0 \ln\rho\, n_\rho \right\} \mathrm{d}l - \left(\frac{1}{\rho}H_{,\rho z}\right)_{,t} \mathrm{d}l.\end{aligned} \tag{5.21}$$

Integrate this relationship along the closed path $\mathscr{L} \subset \mathscr{A}$: on the left hand side $\oint_{\mathscr{L}} \mathrm{d}_t w = 0$ in view of the required continuity of the displacement component; the last term on the right hand side, $\oint_{\mathscr{L}}((1/\rho)H_{,\rho z})_{,t}\mathrm{d}l$, is trivially nil owing to the implicitly assumed regularity of the unknown function H; then the remaining expression is

$$0 = \oint_{\mathscr{L}} \frac{1}{\rho}(\Lambda H)_{,n}\,\mathrm{d}l - c_0^* \oint_{\mathscr{L}} \ln\rho\, n_\rho\,\mathrm{d}l. \tag{5.22}$$

The last line integral along $\mathscr{L}$ can be transformed in a surface integral. Let Int($\mathscr{L}$) be the domain defined as the subset of $\Re^2$ enclosed within the curve $\mathscr{L}$ (Fig. 4); note that Int($\mathscr{L}$) is simply connected in any case and $\boldsymbol{n}$ is its exterior normal. Hence

$$\oint_{\mathscr{L}} \ln\rho n_\rho\,\mathrm{d}l \equiv \int_{\mathrm{Int}(\mathscr{L})} \frac{1}{\rho}\,\mathrm{d}a. \tag{5.23}$$

Writing down Eq. (5.22) for $\mathscr{L} \equiv c_i$, the final form of the *discrete compatibility conditions* (equivalent to the w monodromy) are finally obtained

$$\oint_{c_1} \frac{1}{\rho}(\Lambda H)_{,n}\,\mathrm{d}l = +c_0^* \int_{\mathrm{Int}(c_1)} \frac{1}{\rho}\,\mathrm{d}a, \tag{5.24}$$

$$\oint_{c_1} \frac{1}{\rho}(\Lambda H)_{,n}\,\mathrm{d}l = -c_0^* \int_{\mathrm{Int}(c_i)} \frac{1}{\rho}\,\mathrm{d}a, \quad i = 2,\ldots,N, \tag{5.25}$$

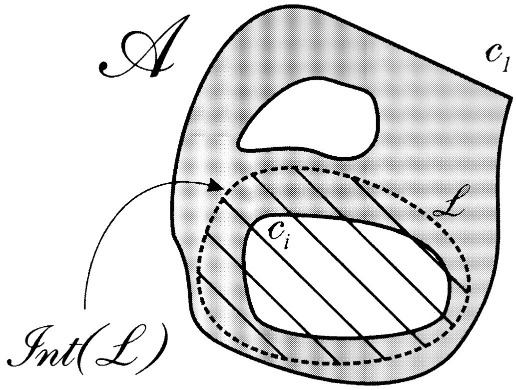

Fig. 4. Definition of $\mathrm{Int}(c_i)$, delimited by the closed curve c_i.

where the minus sign appears in Eq. (5.25) because in these relationships $\boldsymbol{n}$ is again the exterior normal with respect to $\mathscr{A}$, instead of $\mathrm{Int}(c_i)$.

Theorem 2. *Only $N-1$ discrete compatibility conditions are linearly independent.*

Proof. The sum of Eq. (5.24) with Eq. (5.25) gives

$$\int_{\partial\mathscr{A}} \frac{1}{\rho}(\Lambda H)_{,n}\,\mathrm{d}l = c_0^* \int_{\mathscr{A}} \frac{1}{\rho}\,\mathrm{d}a, \tag{5.26}$$

since $\mathscr{A}$ is the complement of $\cup_{i=2}^{N}\mathrm{Int}(c_i)$ with respect to $\mathrm{Int}(c_1)$.

This equation is identically satisfied; in fact, by integrating the governing equation (5.6) multiplied by ρ on the domain $\mathscr{A}$, results in

$$c_0^* \int_{\mathscr{A}} \frac{1}{\rho}\,\mathrm{d}a = \int_{\mathscr{A}} \left\{ \frac{1}{\rho}\left[(\Lambda H)_{,\rho}\right]_{,\rho} + \left[\frac{1}{\rho}(\Lambda H)_{,z}\right]_{,z} \right\} \mathrm{d}a \equiv \int_{\partial\mathscr{A}} \frac{1}{\rho}(\Lambda H)_{,n}\,\mathrm{d}l. \tag{5.27}$$

Thus Eqs. (5.24) and (5.25) are linearly dependent.

Furthermore, the proof shows that any one of the N discrete compatibility conditions can be eliminated. □

5.4. Equation in ψ

Consider first that, from Eq. (5.1), function ψ can be expressed using the Λ_ρ part of the operator Λ as

$$\psi = -\Lambda_\rho H; \tag{5.28}$$

secondarily, it is easy to see that Λ and Λ_z commute: $\Lambda\Lambda_z \equiv \Lambda_z\Lambda$; finally, consider the following chain of identities:

$$\Lambda H \equiv \Lambda_\rho H + \Lambda_z H = -\psi + \Lambda_z H, \tag{5.29}$$

$$\Lambda\Lambda H = -\Lambda\psi + \Lambda\Lambda_z H = -\Lambda\psi + \Lambda_z\Lambda H, \tag{5.30}$$

$$\Lambda(c_0^*) = \Lambda(\Lambda\Lambda H) = -\Lambda\Lambda\psi + \Lambda_z\Lambda\Lambda H = -\Lambda\Lambda\psi + \Lambda_z c_0^*, \tag{5.31}$$

based on Eqs. (5.28) and (5.6).

Thus the required equation in ψ is

$$\Lambda\Lambda\psi = 0. \tag{5.32}$$

Remarks. (1) The expansion (5.4) of the governing equation shows that it is an *elliptic* PDE, with variable coefficients, in the domain $\mathscr{A}$.

(2) The boundary conditions are of a very special kind:

- No essential BCs are assigned;
- Eq. (5.9) is an *oblique degenerate* BC, along a *constant direction* (ρ-axis); it involves the second order derivative of H (*unstable* BC);
- Eq. (5.15), or (5.11), is an oblique degenerate BC involving up to the third order partial derivative of H (*unstable* BC).

From the point of view of functional analysis, a general discussion of the solution existence for a so difficult BVP is not yet available. On the contrary, the degenerate oblique derivative problem for *second order* systems is well studied even in non linear cases (see the masterly Chapter 19 in Mikhlin (1970) in which analytical functions are employed). Alternatively, the second order BVP can be restated as a singular integral equation (Yanushauskas, 1989) or studied within the scope of the pseudo-differential operators theory (Popivanov and Pagachev, 1997).

(3) In this regard, it is useful to anticipate here the conclusion of Section 6 in Part II: the *differential system* governing the bending of a curved beam is *non-variational*, in the sense that the field equation *and* the BCs cannot be derived from any (quadratic) functional of H.

(4) On the other hand, the BVP in the ψ unknown seems to be more tractable. In fact the field equation $\Lambda\Lambda\psi = 0$, with the condition $\psi = \psi_i^0$ on $\partial\mathscr{A}$, is almost a standard problem. Unfortunately, the second boundary condition (2.13), or (5.11), resists reformulation in terms of ψ alone (the question will be resolved with the transformation presented in Section 7).

6. Force resultants

In this paragraph the components of the force resultants, N_φ, V_ρ, V_z, M_ρ, M_z, M_η, are computed to ascertain that the beam is subjected to a bending moment about z (in Part II analogous results are obtained *without* using the *explicit* solution of the equilibrium equation).

It is interesting to anticipate that both the *centroid* (in any sense it could be defined in a purely geometric manner) and a *neutral* "*axis*" do not appear in the results obtained.

6.1. Normal force

The resultant of the σ_φ stress components, acting on the current cross-section $\mathscr{A}_\varphi$, in the direction of its normal is:

$$N_\varphi := \int_{\mathscr{A}} \sigma_\varphi \, \mathrm{d}a = \int_{\mathscr{A}} [(\rho\sigma_\rho)_{,\rho} + \psi_{,zz}] \, \mathrm{d}a = \int_{\partial\mathscr{A}} \rho \left[\sigma_\rho n_\rho + \frac{1}{\rho} \psi_{,z} n_z \right] \mathrm{d}l \equiv 0, \tag{6.1}$$

in which the evaluation (2.10) and the BC (2.13) are used.

The shear forces, V_ρ and V_z, and the moment component M_z vanish identically because $\tau_{\rho\varphi}$ and $\tau_{\varphi z}$ are assumed to be zero.

6.2. Moment component along the ρ-axis

The moment of the normal stresses *with respect to the* ρ-axis, on each cross-section $\mathscr{A}_\varphi$ is:

$$M_\rho := \int_{\mathscr{A}} z\sigma_\varphi \, \mathrm{d}a = \int_{\mathscr{A}} z[(\rho\sigma_\rho)_{,\rho} + (\psi_{,zz})] \, \mathrm{d}a = \int_{\mathscr{A}} [(\rho z\sigma_\rho)_{,\rho} + (z\psi_{,z}) - \psi_{,z}] \, \mathrm{d}a$$
$$= \int_{\partial\mathscr{A}} z\rho \left[\sigma_\rho n_\rho + \frac{1}{\rho}\psi_{,z} n_z\right] \mathrm{d}l + \sum_i \left[\psi_i \int_{c_i} n_z \, \mathrm{d}l\right] \equiv 0, \tag{6.2}$$

where the last equality is obtained using the considerations leading $N_\varphi = 0$ and the BCs (2.12).

6.3. Bending moment

In conclusion, the moment of the normal stresses *with respect to the* z-axis, on each cross-section $\mathscr{A}_\varphi$, is the only force component different from zero; two alternative expressions are provided in the following:

$$M_z := \int_{\mathscr{A}} \rho\sigma_\varphi \, \mathrm{d}a = \int_{\mathscr{A}} \rho[(\rho\sigma_\rho)_{,\rho} + \psi_{,zz}] \, \mathrm{d}a = \int_{\mathscr{A}} [(\rho^2\sigma_\rho)_{,\rho} + (\rho\psi_{,z})_{,z} - \rho\sigma_\rho] \, \mathrm{d}a$$
$$= \int_{\partial\mathscr{A}} \rho^2 \left[\sigma_\rho n_\rho + \frac{1}{\rho}\psi_{,z} n_z\right] \mathrm{d}l - \int_{\mathscr{A}} \rho\sigma_\rho \, \mathrm{d}a = -\int_{\mathscr{A}} \rho\sigma_\rho \, \mathrm{d}a, \tag{6.3}$$

in which the BC (2.13) is used in the integral along $\partial\mathscr{A}$.

The last expression shows that *great influence* of the stress component σ_ρ in exact formulations: it gives M_z formally *in the same way* as σ_φ.

This relationship permits determining the constant c_0 that appears on the right hand of the field equation.

7. The degenerate oblique (unstable) derivative fourth order BVP in a ρ-convex domain

This paragraph presents the solution of the BVP governing the bending of a curved beam. The main outcome is Lemma 5 in which an appropriate integral representation of the H function is introduced to *transform one* degenerate *unstable* condition into an *essential* BC. An elementary solution for a *second order* degenerate BVP in a circle can be found at the end of Chapter V, Section 4, in Bitsadze (1968).

7.1. Incomplete degenerate boundary value problem

Consider first the following *incomplete* degenerate BVP

$$\Lambda\Lambda H(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \mathscr{A}, \tag{7.1}$$

$$\rho\left(\frac{1}{\rho}H_{,\rho}(\mathbf{x})\right)_{,\rho} = g(\mathbf{x}), \quad \forall \mathbf{x} \in \partial\mathscr{A}. \tag{7.2}$$

Notice that this BVP has a very special structure:

1. the field equation is *homogeneous*;
2. only *one* BC is assigned (*incomplete* BVP), $g(\mathbf{x})$ being a regular function defined on the boundary;

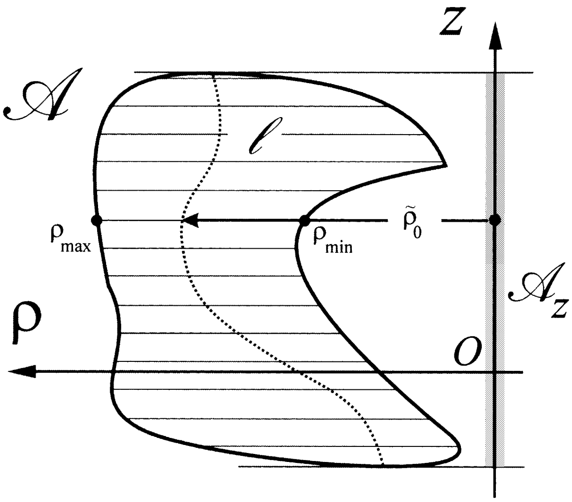

Fig. 5. ρ-convex cross-section domain with a regular interior curve.

3. the BC involves the second order derivative (*unstable* BC);
4. the derivative is prescribed in a *constant* direction that is not normal to the boundary (*oblique* BC); in addition, this direction becomes tangent to the boundary at several points (*degenerate* oblique BC).

7.2. Domain hypotheses

A transformation of this BVP is derived in the following hypotheses on the cross-section domain:

Domain hypothesis 1 (DH1). $\mathscr{A}$ is ρ-*convex*; i.e. every segment parallel to the ρ-axis lies within $\mathscr{A}$ provided that its extreme points belong to $\mathscr{A}$; furthermore, no finite segment parallel to the ρ-axis belongs to the boundary.

Domain hypothesis 2 (DH2). There is sufficiently regular curve ℓ contained in $\mathscr{A}$, $\ell := \{(\rho, z) \in \mathscr{A} \,|\, \rho = \widetilde{\rho}_0(z), z \in \mathscr{A}_z\}$, where $\mathscr{A}_z$ is the projection of $\mathscr{A}$ on the z-axis (see Fig. 5).

From DH1 it follows that $\mathscr{A}$ is *simply connected*; moreover, every line parallel to the ρ-axis intersects the boundary $\partial\mathscr{A}$ exactly at two points (with abscissa $\rho_{\min}(z)$ and $\rho_{\max}(z)$, respectively) such that $\rho_{\min}(z) \leqslant \widetilde{\rho}_0(z) \leqslant \rho_{\max}(z)$.

7.3. Preliminary results

Lemma 3. *The operator* $\Lambda_\rho := \rho\partial_\rho(1/\rho)\partial_\rho$ *admits an explicit inverse, in the following sense*:

$$\frac{1}{2}\int_{\rho_0}^{\rho} \frac{\rho^2 - \xi^2}{\xi} \Lambda_\rho u(\xi)\,\mathrm{d}\xi \equiv u(\rho) - \left[u(\rho_0) + \frac{1}{2}\frac{\rho^2 - \rho_0^2}{\rho_0} u_{,\rho}(\rho_0)\right]. \tag{7.3}$$

Proof. Consider the ODE $\Lambda_\rho u(\rho) = v(\rho)$; by a quadrature

$$\frac{1}{\rho}u_{,\rho} = \int_{\rho_0}^{\rho} \frac{1}{\xi} v(\xi)\,\mathrm{d}\xi + \frac{1}{\rho_0} u_{,\rho}(\rho_0) \tag{7.4}$$

is obtained; an integration by parts of the first term on the right hand side of this relationship gives

$$\int_{\rho_0}^{\rho} \left\{ \eta \int_{\rho_0}^{\eta} \frac{1}{\xi} v(\xi)\,\mathrm{d}\xi \right\} \mathrm{d}\eta = \left[\frac{1}{2}\eta^2 \int_{\rho_0}^{\eta} \frac{1}{\xi} v(\xi)\,\mathrm{d}\xi \right]_{\rho_0}^{\rho} - \int_{\rho_0}^{\rho} \frac{1}{2}\eta v(\eta)\,\mathrm{d}\eta = \frac{1}{2}\int_{\rho_0}^{\rho} \frac{\rho^2 - \xi^2}{\xi} v(\xi)\,\mathrm{d}\xi \tag{7.5}$$

so that the final form of the solution can be written as

$$u(\rho) = u(\rho_0) + \frac{1}{2}\frac{\rho^2 - \rho_0^2}{\rho_0} u_{,\rho}(\rho_0) + \frac{1}{2}\int_{\rho_0}^{\rho} \frac{\rho^2 - \xi^2}{\xi} v(\xi)\,\mathrm{d}\xi, \tag{7.6}$$

from which the lemma can be obtained. □

Lemma 4. *Let $\upsilon(\rho, z)$ satisfy the PDE $\Lambda\Lambda\upsilon = 0$ in a ρ-convex domain. Then*

$$\frac{1}{2}\int_{\rho_0}^{\rho} \frac{\rho^2 - \xi^2}{\xi} \upsilon_{,zzzz}(\rho, z)\,\mathrm{d}\xi = -\Lambda_\rho(\Lambda_\rho + 2\Lambda_z)\upsilon(\rho, z) + (\Lambda_\rho + 2\Lambda_z)\upsilon(\rho_0, z) + \frac{1}{2}\frac{\rho^2 - \rho_0^2}{\rho_0}\partial_\rho(\Lambda_\rho + 2\Lambda_z)\upsilon(\rho_0, z), \ \textit{in } \mathscr{A}. \tag{7.7}$$

Proof. Since Λ and Λ_z commute, from definitions (5.7) the identity $\Lambda\Lambda = \Lambda_\rho\Lambda_\rho + 2\Lambda_\rho\Lambda_z + \Lambda_z\Lambda_z = \Lambda_\rho(\Lambda_\rho + 2\Lambda_z) + \Lambda_z\Lambda_z$ holds; therefore if $\Lambda\Lambda\upsilon = 0$ then $\Lambda_z\Lambda_z\upsilon \equiv -\Lambda_\rho(\Lambda_\rho + 2\Lambda_z)\upsilon$. Using this result, the integral on the left-hand side of Eq. (7.7) can be written as

$$\begin{aligned}\frac{1}{2}\int_{\rho_0}^{\rho} \frac{\rho^2 - \xi^2}{\xi} \upsilon_{,zzzz}(\rho, z)\,\mathrm{d}\xi &= -\frac{1}{2}\int_{\rho_0}^{\rho} \frac{\rho^2 - \xi^2}{\xi} \Lambda_\rho(\Lambda_\rho + 2\Lambda_z)\upsilon(\rho, z)\,\mathrm{d}\xi \\ &= -(\Lambda_\rho + 2\Lambda_z)\upsilon(\rho, z) + (\Lambda_\rho + 2\Lambda_z)\upsilon(\rho_0, z) + \frac{1}{2}\frac{\rho^2 - \xi^2}{\xi}\partial_\rho(\Lambda_\rho + 2\Lambda_z)\upsilon(\rho_0, z)\end{aligned} \tag{7.8}$$

in which the second equality is obtained by applying Lemma 3. □

Lemma 5. *Let $\mathscr{A}$ satisfy the hypotheses DH*1 *and* 2. *Then a necessary and sufficient condition for the function $H(\boldsymbol{x})$ to be the solution of the incomplete degenerate BVP* (7.1) *and* (7.2) *is that the following integral representation holds*

$$H(\rho, z) := \frac{1}{2}\int_{\widetilde{\rho_0}(z)}^{\rho} \frac{\rho^2 - \xi^2}{\xi} \upsilon(\xi, z)\,\mathrm{d}\xi + \rho^2\alpha_{0\ldots4}(z) + \beta_{0\ldots4}(z), \tag{7.9}$$

where $\upsilon(\rho, z)$ satisfies the following incomplete BVP:

$$\Lambda\Lambda\upsilon(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \mathscr{A}, \tag{7.10}$$

$$\upsilon(\mathbf{x}) = g(\mathbf{x}), \quad \forall \mathbf{x} \in \partial\mathscr{A}, \tag{7.11}$$

and functions $\alpha_{0\ldots4}(z) := \sum_{i=0}^{4}\alpha_i(z)$ and $\beta_{0\ldots4}(z) := \sum_{i=0}^{4}\alpha_i(z)$ are solutions of the following ODEs on the manifold $\mathscr{A}_z$:

$$\frac{\mathrm{d}^{i+1}}{\mathrm{d}z^{i+1}}\alpha_i(z) - \frac{1}{2}[\widetilde{\rho_0}(z)]_{,z}\frac{1}{\widetilde{\rho_0}(z)}\frac{\partial^i}{\partial z^i}\upsilon[\rho, z]_{\rho=\widetilde{\rho_0}(z)} = 0, \quad i = 0\ldots3, \tag{7.12}$$

$$\rho_0^2 \frac{\mathrm{d}^4}{\mathrm{d}z^4} \alpha_4(z) = -\frac{1}{2} \rho_0 \partial_\rho (\Lambda_\rho + 2\Lambda_z) \upsilon[\rho, z]_{\rho=\widetilde{\rho}_0(z)}, \tag{7.13}$$

$$\frac{\mathrm{d}^{i+1}}{\mathrm{d}z^{i+1}} \beta_i(z) + \frac{1}{2} [\widetilde{\rho}_0(z)]_{,z} \widetilde{\rho}_0(z) \frac{\partial^i}{\partial z^i} \upsilon[\rho, z]_{\rho=\widetilde{\rho}_0(z)} = 0, \quad i = 0 \ldots 3, \tag{7.14}$$

$$\frac{\mathrm{d}^4}{\mathrm{d}z^4} \beta_4(z) = -(\Lambda_\rho + 2\Lambda_z) \upsilon(\rho_0, z) + \frac{1}{2} \rho_0 \partial_\rho (\Lambda_\rho + 2\Lambda_z) \upsilon[\rho, z]_{\rho=\tilde{\rho}_0(z)}. \tag{7.15}$$

Proof. (Sufficiency). From Lemma 3 and the definition of H (7.9), the following identities hold

$$\Lambda_\rho H = \upsilon, \tag{7.16}$$

$$\Lambda_\rho \Lambda_z H = \Lambda_z \upsilon. \tag{7.17}$$

The partial derivatives of H with respect to z must be computed step by step as shown below:

$$H_{,z} = \frac{1}{2} \int_{\tilde{\rho}_0(z)}^{\rho} \frac{\rho^2 - \xi^2}{\xi} \upsilon_{,z}(\xi, z) \, \mathrm{d}\xi - \frac{1}{2} [\tilde{\rho}_0(z)]_{,z} \frac{\rho^2 - \tilde{\rho}_0(z)}{\tilde{\rho}_0(z)} \upsilon[\tilde{\rho}_0(z), z] + \rho^2 \alpha_{0\ldots4,z}(z) + \beta_{0\ldots4,z}(z). \tag{7.18}$$

To simplify the subsequent development, the (undetermined) functions α_0 and β_0 are used to eliminate the second addendum of this equation, i.e. they are constrained to identically satisfy the following ODEs on $\mathscr{A}_Z$:

$$\alpha_{1,z}(z) - \frac{1}{2} [\tilde{\rho}_0(z)]_{,z} \frac{1}{\tilde{\rho}_0(z)} \upsilon[\tilde{\rho}_0(z), z] = 0, \tag{7.19}$$

$$\beta_{1,z}(z) + \frac{1}{2} [\tilde{\rho}_0(z)]_{,z} \tilde{\rho}_0(z) \upsilon[\tilde{\rho}_0(z), z] = 0. \tag{7.20}$$

Consequently $H_{,z}$ reduces to

$$H_{,z} = \frac{1}{2} \int_{\tilde{\rho}_0(z)}^{\rho} \frac{\rho^2 - \xi^2}{\xi} \upsilon_{,z}(\xi, z) \mathrm{d}\xi + \rho^2 \alpha_{1\ldots4,z}(z) + \beta_{1\ldots4,z}(z), \tag{7.21}$$

that maintains the same formal structure H possesses in Eq. (7.9). By reiterating the process, it is easy to realize that $\Lambda_z \Lambda_z H$ can be written as

$$\Lambda_z \Lambda_z H = \frac{1}{2} \int_{\tilde{\rho}_0(z)}^{\rho} \frac{\rho^2 - \xi^2}{\xi} \Lambda_z \Lambda_z \upsilon(\xi, z) \, \mathrm{d}\xi + \rho^2 \alpha_{4,z}(z) + \beta_{4,z}(z), \tag{7.22}$$

assuming that the α_i and β_i functions satisfy the ODEs

$$\frac{\mathrm{d}^{i+1}}{\mathrm{d}z^{i+1}} \alpha_i(z) - \frac{1}{2} [\widetilde{\rho}_0(z)]_{,z} \frac{1}{\widetilde{\rho}_0(z)} \frac{\partial^i}{\partial z^i} \upsilon[\rho, z]_{\rho=\widetilde{\rho}_0(z)} = 0, \quad i = 0, \ldots, 3, \tag{7.23}$$

$$\frac{\mathrm{d}^{i+1}}{\mathrm{d}z^{i+1}} \beta_i(z) + \frac{1}{2} [\widetilde{\rho}_0(z)]_{,z} \widetilde{\rho}_0(z) \frac{\partial^i}{\partial z^i} \upsilon[\rho, z]_{\rho=\widetilde{\rho}_0(z)} = 0, \quad i = 0, \ldots, 3, \tag{7.24}$$

It is worthy of mention that the *initial values* of these functions are quite unessential and could therefore be ignored.

Assuming that function υ obeys the hypothesis of Lemma 5, $\Lambda\Lambda\upsilon = 0$, Eq. (7.22) becomes

$$\Lambda_z\Lambda_z H = -\Lambda_\rho(\Lambda_\rho + 2\Lambda_z)\upsilon(\rho, z) + (\Lambda_\rho + 2\Lambda_z)\upsilon[\rho, z]_{\rho=\tilde{\rho}_0(z)} + \frac{1}{2}\frac{\rho^2 - \rho_0^2}{\rho_0}\partial_\rho(\Lambda_\rho + 2\Lambda_z)\upsilon[\rho, z]_{\rho=\tilde{\rho}_0(z)} + \rho^2\frac{d^4}{dz^4}\alpha_4(z) + \frac{d^4}{dz^4}\beta_4(z). \tag{7.25}$$

According to the previous adopted strategy this relationship can be simplified in

$$\Lambda_z\Lambda_z H = -\Lambda_\rho(\Lambda_\rho + 2\Lambda_z)\upsilon(\rho, z) \tag{7.26}$$

provided that the function α_4 and β_4 fulfill the ODE on $\mathscr{A}_Z$:

$$\rho_0^2\frac{d^4}{dz^4}\alpha_4(z) = -\frac{1}{2}\rho_0\partial_\rho(\Lambda_\rho + 2\Lambda_z)\upsilon(\rho_0, z), \tag{7.27}$$

$$\frac{d^4}{dz^4}\beta_4(z) = -(\Lambda_\rho + 2\Lambda_z)\upsilon(\rho_0, z) + \frac{1}{2}\rho_0\partial_\rho(\Lambda_\rho + 2\Lambda_z)\upsilon(\rho_0, z). \tag{7.28}$$

In conclusion, by collecting the results (7.26), (7.16) and (7.17), the desired equation $\Lambda\Lambda H = 0$ is verified.
(***Necessity***): Let $H(\mathbf{x})$ be a solution of BVP (7.1) and (7.2) and υ be defined as

$$\upsilon := \Lambda_\rho H, \tag{7.29}$$

obviously, relationship (7.9) is merely its general integral (accompanied by the relevant consistency initial value conditions).

From the identity $\Lambda\Lambda\Lambda \equiv \Lambda\Lambda\Lambda_\rho + \Lambda_z\Lambda\Lambda$, already used in proving (5.32), Eq. (7.10) is restored

$$0 = \Lambda\Lambda\Lambda H = \Lambda\Lambda\upsilon + \Lambda_z\Lambda\Lambda H = \Lambda\Lambda\upsilon. \tag{7.30}$$

Finally, BC (7.11) is a trivial consequence of the υ definition and Eq. (7.2). □

7.3.1. Identification of υ as the ψ potential function

A review of the proof of Lemma 5 shows that $(-\upsilon)$ can be *identified with the potential stress function* ψ defined by Eq. (5.1) and governed by the field equation (5.32), with ψ_i^0 on $\partial\mathscr{A}$.

7.4. Solution of the degenerate boundary value problem governing curved beams in bending

Owing to the very special form of the degenerate oblique BC appearing in the BVP (5.6), (5.9)–(5.11) in the H function, the question of existence of its solution is reduced to a non-oblique BVP in the potential function ψ.

Theorem 3. *The degenerate oblique unstable BVP governing the uniform bending of curved beams,*

$$\Lambda\Lambda H = c_0^*, \quad \forall\mathbf{x} \in \mathscr{A}, \tag{7.31}$$

$$\rho\left(\frac{1}{\rho}H_{,\rho}\right)_{,\rho} = 0, \quad \forall\mathbf{x} \in \partial\mathscr{A}, \tag{7.32}$$

$$\left[(\nu - 1)H_{,zz} + \rho H_{,\rho zz} + \rho\sigma_\rho^{\mathrm{P}}\right]n_\rho - \rho\left(\frac{1}{\rho}H_{,\rho}\right)_{,\rho z} n_z = 0, \quad \forall\mathbf{x} \in \partial\mathscr{A}, \tag{7.33}$$

has the following solution

$$H(\rho,z) = \frac{1}{2}c_0^* z^4 - \frac{1}{2}\int_{\tilde{\rho}_0(z)}^{\rho} \frac{\rho^2-\xi^2}{\xi}\psi(\xi,z)\,\mathrm{d}\xi + \rho^2\alpha_{0\ldots4}(z) + \beta_{0\ldots4}(z), \tag{7.34}$$

in which ψ *satisfies the following* BVP

$$\Lambda\Lambda\psi(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \mathscr{A}, \tag{7.35}$$

$$\psi(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \partial\mathscr{A}, \tag{7.36}$$

$$(\rho H[\psi]_{,\rho zz} + (\nu-1)H[\psi]_{,zz} + \rho\sigma_\rho^{\mathrm{P}})n_\rho - \rho\left(\frac{1}{\rho}H[\psi]_{,\rho}\right)_{,\rho z} n_z = 0, \quad \forall \mathbf{x} \in \partial\mathscr{A}, \tag{7.37}$$

and $\alpha_i(z)$, $\beta_i(z)$ *are solution of ODEs* (7.12)–(7.15), *in which* υ *must be replaced by* $(-\psi)$.

Please note that the last BC is actually a boundary integro-differential equation in ψ.

8. A model case: the rectangular cross-section

The possibility of finding a solution for the degenerate unstable BVP derived above *still remains open*, at least in the ρ-convex domain. To this end, in this last paragraph a particularly simple cross-section is examined, as a model case: the minimal algebra involved permits showing that the problem can be reduced to an (almost) standard BVP, a solution for which can be found, e.g. by a finite difference technique.

8.1. Rectangular cross-section

Despite the fact that the hypothesis of ρ-convex cross-sections does not permit a boundary parallel to the ρ-axis, when these segments appear *only at the extreme* ordinates of the cross-section, the analytical problems can be bypassed by ad hoc considerations.

Consider a rectangular cross-section whose sides parallel to the z-axis are located at ρ_0 and $\rho_1 > \rho_0$, respectively, while the others have ordinate $\pm z_0$. The curve ℓ, introduced above, is suitably chosen coinciding with the side at minimal abscissa: $\tilde{\rho}_0(z) \equiv \rho_0$

Whenever $\tilde{\rho}_0(z) =$ *constant* occurs, the relationships presented in Theorem 3 are greatly simplified because $\alpha_{0\ldots4}(z) \equiv \alpha_4(z), \beta_{0\ldots4}(z) \equiv \beta_4(z)$ and Eqs. (7.12)–(7.14) disappear; thus

$$H(\rho,z) = \frac{1}{2}c_0^* z^4 - \frac{1}{2}\int_{\rho_0}^{\rho} \frac{\rho^2-\xi^2}{\xi}\psi(\xi,z)\mathrm{d}\xi + \rho^2\alpha_4(z) + \beta_4(z). \tag{8.1}$$

The first BC (Eq. (7.36)) simply states $\psi = 0$ on $\partial\mathscr{A}$.

8.1.1. Third boundary condition: horizontal sides

When $n_\rho = 0$, Eq. (7.37) reduces to the only term $\rho((1/\rho)H_{,\rho})_{,\rho z} = 0$ that, owing to the fundamental result (7.16), $\Lambda_\rho H = -\psi$, can be written as

$$\psi_{,z} = 0, \quad \{\mathbf{x} \in [\rho_0, \rho_1], z = \pm z_0\}. \tag{8.2}$$

8.1.2. Consistency conditions

Since $\psi = 0$ on the boundary, $\Lambda_z\psi(\rho,z) \equiv 0$ on the vertical sides (but $\partial_\rho\Lambda_z\psi(\rho,z) \neq 0$, of course). Consequently the consistency conditions (7.13) and (7.15) at $\rho = \rho_0$ can be slightly simplified as

$$\rho_0^2\alpha_4^{\mathrm{iv}}(z) = +\tfrac{1}{2}\rho_0\partial_\rho(\Lambda_\rho + 2\Lambda_z)\psi(\rho_0,z), \tag{8.3}$$

$$\beta_4^{\text{iv}}(z) = +\Lambda_\rho\psi(\rho_0, z) - \tfrac{1}{2}\rho_0\partial_\rho(\Lambda_\rho + 2\Lambda_z)\psi(\rho_0, z), \tag{8.4}$$

in which the derivatives of $\alpha_4(z)$ and $\beta_4(z)$ are denoted through apexes, e.g. $\mathrm{d}^4/\mathrm{d}z^4 \equiv (\ )^{\text{iv}}$, without ambiguity.

From these equations it is immediate to derive the following relationship (always true when the curve ℓ is the straight line $\rho = \rho_0$):

$$\rho_0^2\alpha_4^{\text{iv}}(z) + \beta_4^{\text{iv}}(z) = \Lambda_\rho\psi(\rho_0, z). \tag{8.5}$$

8.1.3. Third boundary condition: side $\rho = \rho_0$

In order to simplify the expressions, the particular solution σ_ρ^{P} has been chosen in such a way that $\sigma_\rho^{\mathrm{P}}(\rho_0) = 0$ and $\sigma_\rho^{\mathrm{P}}(\rho_1) = 0$, as explained in Appendix B.

On the vertical sides $n_z = 0$, so that Eq. (7.37) reduces to

$$\rho H_{,\rho zz} + (\nu - 1)H_{,zz} + \rho\sigma_\rho^{\mathrm{P}} = 0 \tag{8.6}$$

Since

$$H_{,\rho} = -\int_{\rho_0}^{\rho} \frac{\rho}{\xi}\psi(\xi, z)\mathrm{d}\xi + 2\rho\alpha_4(z), \tag{8.7}$$

the BC (8.6), at $\rho = \rho_0$, simply gives

$$(1+\nu)\rho_0^2\alpha_4'' - (1-\nu)\beta_4'' = \tfrac{1}{2}(1-\nu)c_0^* z^2. \tag{8.8}$$

The consistency conditions (8.3) and (8.4) can be employed to remove the unknown function from this equation (after a double derivative):

$$\rho_0\partial_\rho(\Lambda_\rho + 2\Lambda_z)\psi(\rho_0, z) - (1-\nu)\Lambda_\rho\psi(\rho_0, z) = (1-\nu)c_0^*. \tag{8.9}$$

The explicit expressions of α_4^{iv} and β_4^{iv} will be useful in the sequel: from the double derivative of Eq. (8.8) and the consistency condition (8.5)

$$\rho_0^2\alpha_4^{\text{iv}} = \frac{1}{2}(1-\nu)[c_0^* + \Lambda_\rho\psi(\rho_0, z)], \tag{8.10}$$

$$\beta_4^{\text{vi}} = -\frac{1}{2} + (1-\nu)\left[c_0^* - \frac{(1+\nu)}{(1-\nu)}\Lambda_\rho\psi(\rho_0, z)\right] \tag{8.11}$$

can be obtained.

8.1.4. Third boundary condition: side $\rho = \rho_1$

The condition (8.6) evaluated at $\rho = \rho_1$, after rearranging the terms under integral, becomes

$$\frac{1}{2}\int_{\rho_0}^{\rho} \frac{(1+\nu)\rho_1^2 + (1-\nu)\xi^2}{2\xi}\psi_{,zz}(\xi, z)\mathrm{d}\xi - (1+\nu)\rho_0^2\alpha_4''(z)\left(\frac{\rho_1^2}{\rho_0^2} - 1\right) = 0. \tag{8.12}$$

This (integro-differential) equation is quite cumbersome to handle; therefore, according to what was carried out for the previous case, the elimination of $\alpha_4''(z)$ would give a BC involving only ψ.

After a double derivative with respect to z, this equation becomes

$$-\frac{1}{2}\int_{\rho_0}^{\rho_1} \frac{(1+\nu)\rho_1^2 + (1-\nu)\xi^2}{2\xi}\Lambda_\rho(\Lambda_\rho + 2\Lambda_z)\psi(\xi, z)\mathrm{d}\xi - \mu_2[c_0^* + \Lambda_\rho\psi(\rho_0, z)] = 0 \tag{8.13}$$

in which $\Lambda\Lambda\psi = 0$ is used to write $\psi_{,zzzz}$ as $-\Lambda_\rho(\Lambda_\rho + 2\Lambda_z)\psi$ and Eq. (8.10) to remove α_4^{iv}; finally $\mu_2 := \frac{1}{2}(1-\nu^2)(\rho_1^2/\rho_0^2 - 1)$.

Carrying out transformations similar to that used in the proof of Lemma 4, this relationship is converted into a *two-point* boundary condition. For this purpose, the following result is required.

Lemma 6.

$$\int_{\rho_0}^{\rho_1} \frac{(1+\nu)\rho_1^2 + (1-\nu)\rho^2}{2\rho} \Lambda_\rho \upsilon(\rho)\,\mathrm{d}\rho \equiv \rho_1 \upsilon_{,\rho}(\rho_1) - \frac{1}{2}\left[(1+\nu)\frac{\rho_1^2}{\rho_0^2} + 1 - \nu\right]\rho_0 \upsilon_{,\rho}(\rho_0) - (1-\nu) \times [\upsilon(\rho_1) - \upsilon(\rho_0)]. \tag{8.14}$$

Proof. Since it is similar to that of Lemma 3 it is omitted. □

In conclusion, applying this result with $\upsilon(\rho) := (\Lambda_\rho + 2\Lambda_z)\psi$ to Eq. (8.13), the anticipated form of the BC at $\rho = \rho_1$ is obtained as

$$\begin{aligned} &-\rho_1\partial_\rho(\Lambda_\rho + 2\Lambda_z)\psi(\rho_1, z) + (1-\nu)\Lambda_\rho\psi(\rho_1, z) + [(1-\nu)(\mu_1 - 1) - \mu_2]\Lambda_\rho\psi(\rho_0, z) \\ &\quad = [\mu_2 - (1-\nu)\mu_1]c_0^*, \end{aligned} \tag{8.15}$$

where $\mu_1 := \frac{1}{2}[(1+\nu)(\rho_1^2/\rho_0^2) + 1 - \nu]$.

This is a *two-point* boundary condition involving up to the third derivative of ψ.

In the case of a trapezoidal cross-section (ρ_1 is a function of z) the last boundary condition remains in the form of an integro–differential equation.

9. Conclusion

In this first paper, a coherent theory of the bending problem in a circular curved beam, having a large radius of curvature with respect to its width, is presented.

The cross-section is multi-connected.

The 3D (linearly isotropic and homogeneous) elastic problem is solved assuming the stress tensor as the unknown and by exactly satisfying the field compatibility equations.

The governing BVP is a *fourth-order* elliptic (variable coefficients) differential system with two degenerate unstable BC.

When the cross-section domain is assumed to be ρ-convex, the problem is reduced to a simpler form. An explicit formulation is derived for the rectangular cross-section with sides parallel to the axes.

The closing remarks, on all the issues arising regarding the bending of a curved beam, are delayed at the end of Part II in order to achieve a global vision of the problem and outline possible enhancements.

Appendix A. Compatibility equations

The following compatibility equations in cylindrical co-ordinates are derived from Reismann and Pawlik (1980).

$$\frac{2}{\rho}\varepsilon_{\varphi z,\varphi z} - \frac{1}{\rho^2}\varepsilon_{z,\varphi\varphi} - \varepsilon_{\varphi,zz} + \frac{2}{\rho}\varepsilon_{\rho z,z} - \frac{1}{\rho}\varepsilon_{z,\rho} = 0, \tag{A.1}$$

$$2\varepsilon_{\rho z,\rho z} - \varepsilon_{\rho,zz} - \varepsilon_{z,\rho\rho} = 0, \tag{A.2}$$

$$\frac{2}{\rho}\varepsilon_{\rho\varphi,\rho\varphi} - \varepsilon_{\varphi,\rho\rho} - \frac{1}{\rho^2}\varepsilon_{\rho,\varphi\varphi} + \frac{1}{\rho}\varepsilon_{\rho,\rho} + \frac{2}{\rho^2}\varepsilon_{\rho\varphi,\varphi} - \frac{2}{\rho}\varepsilon_{\varphi,\rho} = 0, \tag{A.3}$$

$$\frac{1}{\rho}\varepsilon_{z,\rho\varphi} - \left[\frac{1}{\rho}\varepsilon_{\rho z,\varphi} + \varepsilon_{\varphi z,\rho} - \varepsilon_{\rho\varphi,z}\right]_{,z} + \frac{1}{\rho}\varepsilon_{z\varphi,z} - \frac{1}{\rho^2}\varepsilon_{z,\varphi} = 0, \tag{A.4}$$

$$\frac{1}{\rho}\varepsilon_{\rho,\varphi z} - \left[\varepsilon_{\rho\varphi,z} + \frac{1}{\rho}\varepsilon_{\rho z,\varphi} - \varepsilon_{\varphi z,\rho}\right]_{,\rho} - \frac{2}{\rho}\varepsilon_{\rho\varphi,z} + \frac{1}{\rho}\varepsilon_{z\varphi,\rho} - \frac{1}{\rho^2}\varepsilon_{z\varphi} = 0, \tag{A.5}$$

$$\varepsilon_{\varphi,\rho z} - \frac{1}{\rho}\left[\varepsilon_{\varphi z,\rho} + \varepsilon_{\rho\varphi,z} - \frac{1}{\rho}\varepsilon_{\rho z,\varphi}\right]_{,\varphi} - \frac{1}{\rho^2}\varepsilon_{\varphi z,\varphi} - \frac{1}{\rho}(\varepsilon_\rho - \varepsilon_\varphi)_{,z} = 0. \tag{A.6}$$

It is worth noticing that the explicit use of these equations is very rare in actual applications; furthermore a number of mistakes are present in several handbooks (Malvern, 1969; Zyczkowski, 1981).

Appendix B. General integral of Eq. (4.3)

The PDE (4.3) is of the form

$$\left(\frac{1}{\rho}A_{,\rho}\right)_{,\rho} + \frac{1}{\rho}B_{,zz} = c_0\frac{1}{\rho}, \tag{B.1}$$

where $A := (1/\rho)(\rho^2\sigma_\rho + v\psi)_{,\rho}$ and $B := (\rho\psi)_{,\rho} + v\psi$.

B.1. Solution of the homogeneous PDE

Eq. (B.1) is a condition for the existence of a function $F^1(\rho, z)$ (assumed to be monodrome in domain $\mathscr{A}$) such that

$$\frac{1}{\rho}A_{,\rho} = \frac{1}{\rho}F^1_{,z}, \tag{B.2}$$

$$\frac{1}{\rho}B_{,z} = -\frac{1}{\rho}F^1_{,\rho}. \tag{B.3}$$

In the same way, Eq. (B.2) gives

$$A = F^2_{,z}, \tag{B.4}$$

$$F^1 = F^2_{,\rho}. \tag{B.5}$$

while Eq. (B.3), written using the last result as $B_{,z} = -(F^2_{,\rho})_{,\rho}$, gives

$$\frac{1}{\rho}B = \left(\frac{1}{\rho}F^3\right)_{,\rho}, \tag{B.6}$$

$$\frac{1}{\rho}F^2_{,\rho} = -\frac{1}{\rho}F^3_{,z}. \tag{B.7}$$

Finally from the former equation

$$F^2 = F^4_{,z}, \tag{B.8}$$

$$F^3 = -F^4_{,\rho} \tag{B.9}$$

is obtained. Thus, by substituting these results in Eqs. (B.4) and (B.6) the solution can be written as

$$A = (F^4_{,z})_{,z}, \tag{B.10}$$

$$B = -\rho\left(\frac{1}{\rho}F^4_{,\rho}\right)_{,\rho}. \tag{B.11}$$

B.2. Particular solution

A *particular solution* of Eq. (4.3) is easily determined by ignoring ψ^{P} and regarding $\sigma^{\mathrm{P}}_{\rho}$ as a function of ρ only (the PDE becomes an ODE):

$$\sigma^{\mathrm{P}}_{\rho}(\rho) := \frac{1}{4}c_0(2\ln\rho - 1) + c_1 + c_2\frac{1}{\rho^2}. \tag{B.12}$$

The constants c_1 and c_2 are arbitrary, of course, and therefore can be assumed to be nil.

It is however interesting to note that they can be employed to assign arbitrary values to σ_ρ at distances ρ_0 and ρ_1. This is what occurs in the well-known Golovin solution of bending in circular beams with rectangular *thin* cross-section *in plane state* of stress, where $\sigma_\rho = 0$ at the inner and outer surfaces:

$$c_1 = \frac{1}{4}c_0\left(1 - 2\frac{\rho_0^2\ln\rho_0 - \rho_1^2\ln\rho_1}{\rho_0^2 - \rho_1^2}\right), \tag{B.13}$$

$$c_2 = \frac{1}{2}c_0\frac{\rho_0^2\rho_1^2}{\rho_0^2 - \rho_1^2}\ln\frac{\rho_0}{\rho_1}. \tag{B.14}$$

Appendix C. General integral of Eq. (4.5)

Eq. (4.5), rewritten as,

$$\psi_{,\rho} + (1+v)\frac{1}{\rho}\psi = -\left(\frac{1}{\rho}F_{,\rho}\right)_{,\rho}, \tag{C.1}$$

can be regarded as an ODE in ρ, in which z is reckoned as a parameter. The solution of its *associate homogeneous equation* is quite immediate:

$$\psi^{\mathrm{o}} = a\rho^{-(1+v)}, \tag{C.2}$$

a being a constant.

A *particular solution* is found by the *Lagrange method* in the form $\psi^{\mathrm{P}} = \alpha(\rho)\rho^{-(1+v)}$. The relevant governing ODE is

$$\alpha_{,\rho} = -\rho^{(1+v)}\left(\frac{1}{\rho}F_{,\rho}\right)_{,\rho}. \tag{C.3}$$

A first integration by parts gives

$$\alpha = -\rho^{\nu} F_{,\rho} + (1+\nu) \int \rho^{(\nu-1)} F_{,\rho}\, \mathrm{d}\rho. \tag{C.4}$$

With the position,

$$\phi_{,\rho} := \rho^{(\nu-1)} F_{,\rho} \tag{C.5}$$

in which $\phi(\rho, z)$ is a new unknown function, Eq. (C.4) can be quickly integrated; after some algebra ψ^{P} becomes

$$\psi^{\mathrm{P}} = \rho[\rho^{-(1+\nu)}\phi]_{,\rho}. \tag{C.6}$$

Gathering together all these results, the required solution is finally

$$\psi = [a - \rho\phi_{,\rho} + (1-\nu)\phi]\rho^{-(1+\nu)}, \tag{C.7}$$

$$F_{,\rho} = \rho^{(1-\nu)}\phi_{,\rho}. \tag{C.8}$$

As a concluding remark it can be noted that the constant a can be included in ϕ and therefore it is not reported in Eq. (4.7)

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